Exponential-time quantum algorithms for graph coloring problems

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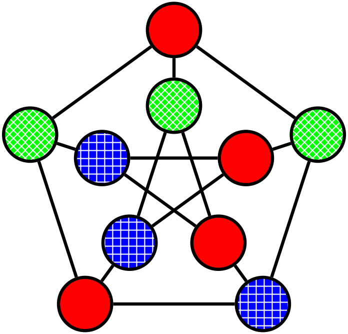
Graph coloring problem



Graph coloring problem:

Input: An undirected graph G

Output: The chromatic number $\chi(G)$ of G

Graph coloring algorithms

- Graph coloring problem is NP-hard.
- Quantum algorithm requires exponential time.
- The fastest classical algorithm requires $O^*(2^n)$ time where n is the number of vertices [Björklund and Husfeldt 2006], [Koivisto 2006]

Theorem (Main theorem)

There is a quantum algorithm using QRAM solving the graph coloring problem with running time $O(1.914^n)$.

First algorithm for graph coloring problem with running time $O((2-\epsilon)^n)$.

Main idea

Grover's search for dynamic programming.

[Ambainis, Balodis, Iraids, Kokainis, Prūsis and Vihrovs 2019]

Previous work

- Grover's algorithm gives quadratic speedup for exaustive search [Grover 1996]

- The exausitive search for travellng salesman problem (TSP) needs $O^*((n-1)!)$ time

- There is a classical algorithm based on dynamic programming with running time $O^*(2^n)$ [Bellman 1962], [Held and Karp 1962]

- Grover's algorithm can be applied to some algorithms based on dynamic programming [Ambainis et al. 2019]

- There is a quantum algorithm for TSP with running time $O(1.728^n)$ [Ambainis et al. 2019]

Dynamic programming for TSP

For $S \subseteq V$ and $v \in S$,

$$A(S, v) := \text{length of the shortest path from 1 to } v \text{ visiting all vertices in } S.$$

Recursive formula for $A(S, v)$:

$$A(\{1\}, 1) = 0$$
$$A(S, v) = \min_{u \in S \setminus \{v\}} \{A(S \setminus \{v\}, u) + \mathrm{dist}(u, v)\}$$

$$\mathrm{OPT}(G) = \min_{v \in V} \{A(V, v) + \mathrm{dist}(v, 1)\}$$

DP needs $O(n^2 2^n)$ operations
[Bellman 1962], [Held and Karp 1962]

QRAM

RAM (random access memory):

$$i \longmapsto m(i)$$

QRAM (quantum random access memory):

$$|i\rangle\,|0\rangle \longmapsto |i\rangle\,|m(i)\rangle$$

$$\sum_i c_i\,|i\rangle\,|0\rangle \longmapsto \sum_i c_i\,|i\rangle\,|m(i)\rangle$$

Grover search for DP

For $S \subseteq V$ and $u, v \in S$,

$$B(S, u, v) := \text{length of the shortest path from } u \text{ to } v \text{ visiting all vertices in } S.$$

$$\mathrm{OPT}(G) = \min_{S \subseteq V,\, |S| = \lceil n/2 \rceil,\, v \in S} \{B(S, 1, v) + B((V \setminus S) \cup \{1, v\}, v, 1)\}$$

For S of size $n/2$,

$$B(S, u, v) = \min_{T \subseteq S \setminus \{v\},\, |T| = \lceil n/4 \rceil,\, w \in T} \{B(T, u, w) + B((S \setminus T) \cup \{w\}, w, v)\}$$

1. For all S satisfying $|S| \le \lceil n/4 \rceil$ and $u, v \in S$, $B(S, u, v)$ is computed by the classical DP and stored to QRAM.
2. Compute $\mathrm{OPT}(G)$ according to the above recurrence.

$$\binom{n}{\frac{n}{4}} + \sqrt{\binom{n}{\frac{n}{2}}\binom{\frac{n}{2}}{\frac{n}{4}}} = O(1.755^n)$$

[Ambainis et al. 2019]

DP for graph coloring

The chromatic number $\chi(G) \geq 2$ satisfies the following recurrence

$$\chi(G) = \min_{S \subseteq V,\, S \notin \{\varnothing, V\}} \chi(G[S]) + \chi(G[V \setminus S]).$$

IF we can assume $|S| = \lfloor n/2 \rfloor$, then the running time $T(n)$ satisfies

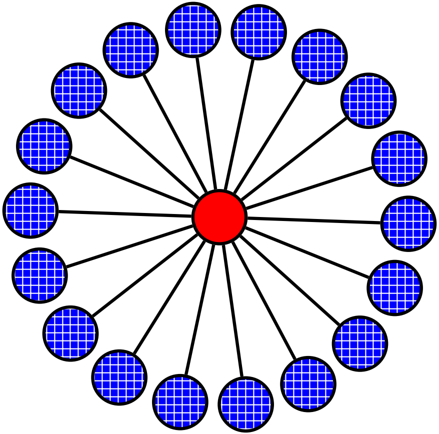
$$T(n) \leq 2^{n/2} T(n/2) \leq 2^{n/2+n/4+n/8+\cdots} \leq 2^n.$$

With the precomputation, we obtain quantum algorithm with running time $O((2-\epsilon)^n)$.

This idea is completely same as [Ambainis et al. 2019].

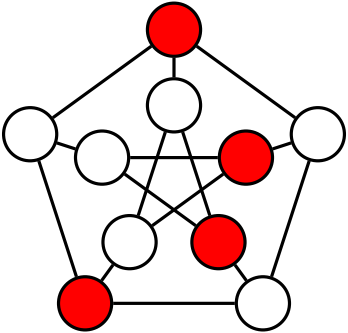
But, this doesn't work.

Unbalanced coloring



Independent set

Independent set is a subset of vertices that are not connected to each other.



Colorable with k colors $\iff$ Vertices can be partitioned into k independent sets

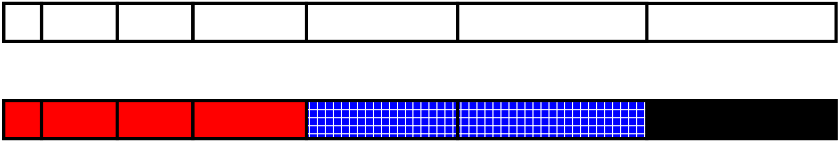
Balanced partition

Lemma

Let $n_1, \ldots, n_k$ be positive integers for $k \geq 1$.

Assume that $n_k \geq n_i$ for all $i \in \{1, 2, \ldots, k-1\}$.

Then, there is $t \in \{1, 2, \ldots, k-2\}$ such that $\sum_{i=1}^{t} n_i \leq n/2$ and $\sum_{i=t+1}^{k-1} n_i \leq n/2$ where $n := \sum_{i=1}^{k} n_i$.



Quantum algorithm for graph coloring

1. For $S \subseteq V$, $|S| \leq n/4$, compute the chromatic number of $G[S]$ and store them to QRAM

2. Let $c = n$ and for each maximal independent set I, partition $V \setminus I$ into S and T such that $|S|$, $|T| \leq n/2$, do following
   - A Compute the chromatic numbers $\chi(G[S])$ and $\chi(G[T])$, recursively (Step 1 is skipped in the recursion)
   - B $c = \min\{c, \chi(G[S]) + \chi(G[T])\}$

3. Output $c + 1$

Quantum algorithm for graph coloring

function $\mathrm{CHR}(G)$
  if G is two colorable **then return** the chromatic number of G
  $\chi[\varnothing] \leftarrow 0$
  for $S \subseteq V$, $S \neq \varnothing$, $|S| \leq \lfloor n/4 \rfloor$ **do** (any order consistent with the inclusion relation)
    $\chi[S] \leftarrow 1 + \min_{I \in \mathrm{MIS}(G[S])}\{\chi[S \setminus I]\}$ [Lawler 1976]
  $c \leftarrow n$
  for $t \in \{1, \ldots, n\}$, $s \in \{\max\{\lceil n/2 \rceil - t, 1\}, \ldots, \lfloor (n-t)/2 \rfloor\}$ **do**
    $a \leftarrow \min_{I \in \mathrm{MIS}(G),\, |I|=t} \min_{T \subseteq V \setminus I,\, |T|=s} (\mathrm{C}(T) + \mathrm{C}(V \setminus I \setminus T))$
    $c \leftarrow \min\{c, a\}$
  return $c + 1$

function $\mathrm{C}(S)$
  if $G[S]$ is two colorable **then return** the chromatic number of $G[S]$
  $c \leftarrow |S|$
  for $t \in \{1, \ldots, |S|\}$, $s \in \{\max\{\lceil |S|/2 \rceil - t, 1\}, \ldots, \lfloor (|S|-t)/2 \rfloor\}$ **do**
    $a \leftarrow \min_{I \in \mathrm{MIS}(G[S]),\, |I|=t} \min_{T \subseteq S \setminus I,\, |T|=s} (\chi[T] + \chi[S \setminus I \setminus T])$
    $c \leftarrow \min\{c, a\}$
  return $c + 1$

Main result

Theorem (Main theorem)

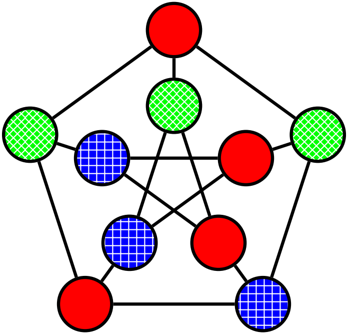
The running time of the quantum algorithm using QRAM for the graph coloring problem has running time

$$O^*\left(\left(2^{37/35}3^{3/7}5^{-9/70}7^{-5/28}\right)^n\right) = O(1.914^n).$$

In fact, there is a faster quantum algorithm although we haven't analyzed yet.

It's first algorithm for graph coloring problem with running time $O((2-\epsilon)^n)$.

Graph k-coloring problem



Graph k-coloring problem:

Input: An undirected graph G

Output: YES if G is k-colorable, and NO otherwise

Previous work

A list of known classical algorithms.

- Graph 3-coloring problem:
  $O(1.3289^n)$ [Beigel and Eppstein 2005]
- Graph 4-coloring problem:
  $O(1.7504^n)$ [Byskov 2004]
  $O(1.7272^n)$ [Fomin, Gaspers, Saurabh 2007]
- Graph 5-coloring problem:
  No classical algorithm with running time $O((2-\epsilon)^n)$ has been known

Our result

Theorem (This work)

There are quantum algorithms not using QRAM for the k-coloring problem with running time $O(\alpha(k)^n)$ where $\alpha(k)$ is shown below, and is less than 2 for $k \leq 20$.

k	$\alpha(k)$
3	1.1528
4	1.3231
5	1.4695
6	1.5261
7	1.6511
8	1.6585

k	$\alpha(k)$
9	1.7460
10	1.7775
11	1.7775
12	1.8246
13	1.8499
14	1.8580

k	$\alpha(k)$
15	1.9303
16	1.9303
17	1.9303
18	1.9366
19	1.9575
20	1.9575

Main Idea:
k-coloring problem can be reduced to k'-coloring problem for $k' < k$ [Byskov 2004].

Summary

- Graph coloring problem can be solved by quantum algorithm using QRAM with running time $O(1.914^n)$
- Graph k-coloring problem can be solved by quantum algorithms not using QRAM with running time $O((2-\epsilon)^n)$ for $k \leq 20$.